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Normal ordering and generalised Stirling numbers

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Abstract. The relation between some normal ordering forms for boson operator functions and generalised Stirling numbers is shown. In particular, the ordering of the operator function $a^k(a^r+N+s)^n$ is obtained for positive integers k, r, n and an arbitrary integer s. This is a generalisation of the recent result of Katriel. Some other normal ordering formulae are presented. It was shown how to obtain antinormal forms from several normal expansions given here.

1. Introduction

The association of combinatorial methods and ordering of functions of non-commuting variables can mutually enrich these fields. The first person to show this was Navon (1973) in the case of ordering of some functions of fermion operators and Katriel (1974) for ordering of boson operator functions. Recently, Mikhailov (1983, to be referred to hereafter as I) using the simplest combinatorial methods for the ordering of functions of boson operators $(a^++a^r)^m$, $(N+a)^m$, $(N+a^2)^m$ obtained explicit expressions. (Here a^+ and a are boson operators, $N=a^+a$.) In the previous case generalised Stirling numbers were introduced and described in detail. After that Katriel (1983) obtained ordering expansion for $(N+a^r)^m$ using only Stirling numbers of the second kind.

There are great opportunities, however, for using generalised Stirling numbers for the ordering. In this paper we show how to generalise the result of Katriel in the case of the ordering of $((a^+)^r+N+s)^n(a^+)^k$ and $a^k(a^r+N+s)^n$ (k, n, r are positive integers, s is arbitrary integer) with the help of these numbers. Besides, we find ordering of the general falling factorial $(lN-kr)_{m,r}$ and, as a particular case, ordering of $((a^+)^rN)^n$. In § 3 we give the antinormal ordering forms for some of the functions described here.

2. Normal ordering formulae

First of all we write the Katriel (1974) ordering formula

$$N^n=\sum_r S(n,r)(a^+)^ra^r \tag{1}$$

which has the combinatorial equivalent

$$x^n=\sum_r S(n,r)(x)_r. \tag{2}$$

Here $S(n, r)$ are Stirling numbers of the second kind, $(x)_r$ is the falling factorial (Riordan 1958). It can easily be seen that correspondence of these formulae rests on a simple relation

$$(a^+)^r a^r = (N)_r = N(N-1)(N-2)\dots(N-r+1). \tag{3}$$

On the other hand there exists an increasing factorial $[x]_r$ which is connected with antinormal monomial

$$a^r(a^+)^r = [N+1]_r = (N+1)(N+2)\dots(N+r). \tag{4}$$

The above relations show that a correspondence between bose operator expressions and combinatorial factorial exists

$$N \to x, \qquad (a^+)^r a^r \to (x)_r, \qquad a^r(a^+)^r \to [x+1]_r. \tag{5}$$

Now we can treat the generating functions for generalised Stirling numbers $S(n, r, k)$ in I as the normal ordering formulae. For example equation (A5) in I

$$x^{n-k}(x)_k = \sum_r S(n, r, k)(x)_r \tag{6}$$

transforms into the operator identity

$$N^{n-k}(a^+)^k = \sum_r S(n, r, k)(a^+)^r a^{r-k} \tag{7}$$

and the exponential generating function for the generalised Stirling numbers (A17) and (A20) in I:

$$\exp(k\alpha)(\exp\alpha - 1)^{r-k}/(r-k)! = \sum_{n=0}^{\infty} S(n+k, r, k)\alpha^n/n! \tag{8}$$

results in another normal ordering expansion

$$\exp(\alpha N)(a^+)^k = \sum_r \exp(k\alpha)(\exp\alpha - 1)^{r-k}(a^+)^r a^{r-k}/(r-k)!. \tag{9}$$

The latter formula transforms into the well known identity (Louisell 1973) at $k=0$. To obtain identity (9) we use formula (7), the rearrangement of the order of summation and expansion (8):

$$\begin{aligned}\exp(\alpha N)(a^+)^k &= \sum_n (\alpha^n/n!) \sum_r S(n+k, r, k)(a^+)^r a^{r-k}\\ &= \sum_r (a^+)^r a^{r-k} \sum_n \alpha^n S(n+k, r, k)/n!\\ &= \sum_r \exp(k\alpha)(\exp\alpha - 1)^{r-k}(a^+)^r a^{r-k}/(r-k)!.\end{aligned}$$

2.1. *Normal ordering of* $A = ((a^+)^r + N + s)^n (a^+)^k$ *and* $B = a^k(a^r + N + s)^n$

Now in accordance with the Witschel (1975) and Katriel (1983) method using generalised Stirling numbers we can generalise equations (5) and (6) of Katriel (1983). In the identity

$$\exp[\alpha(\beta(a^+)^r + N + s)](a^+)^k = \exp\{[\exp(\alpha\beta r) - 1](a^+)^r/r\}\exp[\alpha(N+s)](a^+)^k \tag{10}$$

(here α, β are arbitrary numbers, s is arbitrary integer) which can be obtained with the help of equation (B8) of Kirzhnits (1967) we expand the exponent from the left-hand side and the first exponent from the right-hand side into the Taylor series. For the ordering expansion of $\exp[\alpha(N+s)](a^+)^k$ we use equations (8) and (9):

$$\exp[\alpha(N+s)](a^+)^k = \sum_{r=k}^{\infty}\sum_{q=0}^{\infty} S(q+k+s, r+s, k+s)\alpha^q(a^+)^r a^{r-k}/q!. \qquad (11)$$

Equating coefficients of α^n in the left and right-hand sides of (10) we get finally the normal ordering expansion. Likewise we shall treat the other identity

$$a^k \exp[\alpha(\beta a^r + N + s)] = a^k \exp[\alpha(N+s)]\exp\{[\exp(\alpha\beta r) - 1]a^r/r\} \qquad (12)$$

which can be obtained with the help of equation (B7) of Kirznits (1967). To derive the ordering expansion for $a^k \exp[\alpha(N+s)]$ we use the simple property

$$(a^+)^k a^k \exp[\alpha(N+s)] = (N)_k \exp[\alpha(N+s)] = \exp[\alpha(N+s)]a^{+k}a^k.$$

As a result both normal ordering expansions have proved to be conveniently written in one expression:

$$\left.\begin{matrix}(\beta(a^+)^r + N + s)^n(a^+)^k \\ a^k(\beta a^r + N + s)^n\end{matrix}\right\} = \sum_{i=k}^{n+k}\sum_{j=0}^{n+k-i} C_{ij}^{rnk}(\beta, s)\left\{\begin{matrix}(a^+)^{rj+i}a^{i-k} & (13a) \\ (a^+)^{i-k}a^{rj+i} & (13b)\end{matrix}\right.$$

$$C_{ij}^{rnk}(\beta, s) = \sum_q r^{n-q-j}\beta^{n-q}\binom{n}{q}S(n-q, j)S(q+k+s, i+s, k+s). \qquad (14)$$

For $k=0$, $\beta=1$, $s=1$ equations (13b) and (14) in accordance with property (A29) from I

$$S(n, r, 0) = S(n, r, 1) = S(n, r) \qquad (15)$$

convert into equations (5) and (6) of Katriel (1983).

2.2. Normal ordering of $((a^+)^r N)^n$ and $(lN - kr)_{m,r}$

Making use of the commutator $[N, a^+] = a^+$ one can easily show that $N(a^+)^r = (a^+)^r(N+r)$. Then, by induction we have

$$((a^+)^r N)^n = (a^+)^{rn}N(N+r)(N+2r)\dots(N+(n-1)r)$$
$$\equiv (a^+)^{rn}(N+(n-1)r)_{n,r}. \qquad (16)$$

Thus ordering of $((a^+)^r N)^n$ reduces to the ordering of the falling factorial $(N+(n-1)r)_{n,r}$ or, equivalently, to the expansion of $(x+(n-1)r)_{n,r}$ in a series of $(x)_j$. We shall expand the more general quantity

$$(lx - kr)_{m,r} = (lx - kr)(lx - (k+1)r)(lx - (k+2)r)\dots(lx - (k+m-1)r)$$

which with the help of substitution $t = lx/r$ transforms into usual falling factorial

$$(lx - kr)_{m,r} = r^m(t-k)_m. \qquad (17)$$

By means of the generating function (A7) from I and equation (2) we can obtain

$$(lx - kr)_{n,r} = \sum_j \mathscr{D}_{nj}^{lkr}(x)_j, \qquad (18)$$

$$\mathscr{D}_{nj}^{lkr} = \sum_i l^i r^{n-i} S^{-1}(n+k, i+k, k)S(ij). \qquad (19)$$

Correspondence (5) makes it possible to convert equations (18) and (19) into the ordering expansion of $(lN-kr)_{n,r}$. To derive the ordering expansion of $((a^+)^r N)^n$ we set $l=1$, $k=-n+1$ in equations (18) and (19). Then, we can simplify this expansion with the help of (A10) from I:

$$S^{-1}(n, r, k) = S(n+k-r^{-1}, k-1, n),$$

property (15) and expression of Stirling numbers of the second kind in terms of Stirling numbers of the first kind

$$S(-i, -n) = (-1)^{n+i} s(n, i).$$

The latter can be obtained by comparing recurrence relations for Stirling numbers of the first and second kinds (Riordan 1958). Finally, the ordering expansion is

$$((a^+)^r N)^n = \sum_j \mathscr{D}^r_{nj} (a^+)^{rn+j} a^j, \tag{20}$$

$$\mathscr{D}^r_{nj} = \sum_i (-r)^{n-i} s(ni) S(ij). \tag{21}$$

The numbers $(-1)^n D^r_{nj}$ for $r=1$ convert into Lah numbers (Riordan 1958).

3. Antinormal ordering formulae

Some of the functions just cited can be found easily in the antinormal forms. We are based upon the conservation of boson commutation relations under the following transformations (Biedenharn and Louck 1971)

$$\begin{array}{ccc} \text{(a)} & \text{(b)} & \text{(c)} \\ a^+ \to a & a^+ \to \mathrm{i}a & a^+ \to -a \\ a \to -a^+ & a \to \mathrm{i}a^+ & a \to a^+. \end{array} \tag{22}$$

Shalitin and Tikochinsky (1979) have already used this property for obtaining antinormal expansions.

Applying transformation (22a) to identity (9) we have

$$\exp(\alpha N) a^k = \sum_r \exp[-(k+1)\alpha][1-\exp(-\alpha)]^{r-k} a^r (a^+)^{r-k}/(r-k)!. \tag{23}$$

For $k=0$ we have identity (24) of Shalitin and Tikochinsky (1979). Expanding the coefficients in (23) in accordance with equation (8) and the left-hand side of (23) in a series of α^n and equating coefficients of α^n we find

$$N^{n-k} a^k = \sum_r (-1)^{n+r} S(n+1, r+1, k+1) a^r (a^+)^{r-k}. \tag{24}$$

Similarly we obtain the antinormal expansions from expansions (13) and (14). For transforming ($13a$) it is convenient to use (22a) and for ($13b$) to use (22c). As a result we have

$$\left.\begin{array}{l} (a^+)^k (\beta (a^+)^r + N + s)^n \\ (\beta a^r + N + s)^n a^k \end{array}\right\} = \sum_{i=k}^{n+k} \sum_{j=0}^{n+k-i} B^{rnk}_{ij}(\beta s) \left\{\begin{array}{ll} a^{i-k} (a^+)^{rj+i} & (25a) \\ a^{rj+i} (a^+)^{i-k} & (25b) \end{array}\right.$$

$$B^{rnk}_{ij}(\beta s) = \sum_q (-1)^{i-k-q} r^{n-q-j} \beta^{n-q} \binom{n}{q} S(n-q, j) S(q+k-s+1, i-s+1, k-s+1). \tag{26}$$

4. Conclusion

The ordered operator expansions, including those of this paper, may be used in the quantum description of the interaction of harmonic and various anharmonic oscillators with the quantised fields. Another point of application of the normal and antinormal ordered identities is the determination of matrix elements in the Lie group theory when group operators are expressed in terms of boson operators (Schwinger representation).

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